Monatsh Math (2013) 172:277–291
DOI 10.1007/s00605-013-0501-6

On the completeness of a metric related to the Bergman metric

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Received: 26 October 2012 / Accepted: 19 April 2013 / Published online: 3 May 2013

Abstract We study the completeness of a metric which is related to the Bergman metric of a bounded domain (sometimes called the Burbea metric or Fuks metric). We provide a criterion for its completeness in the spirit of the Kobayashi criterion for the completeness of the Bergman metric. In particular we prove that in hyperconvex domains our metric is complete.

Keywords Bergman metric · Ricci curvature · Completeness · Kobayashi criterion · Hyperconvex domains

Mathematics Subject Classification (2000) 32A36 · 32A25 · 32A40 · 32F45 · 32Q15

1 Introduction

Recall that in a bounded domain $\Omega \subset\subset \mathbb{C}^n$ the Bergman metric is the Kähler metric with metric tensor

$$T_{i\bar{j}}(z) := \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, z), z \in \Omega, i, j = 1, \ldots, n, \tag{1}$$

Communicated by A. Constantin.

The author was partially supported by the NCN grant 2011/01/B/ST1/00879.

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where $K(z,z)$ (or just K for short) is the Bergman kernel (on the diagonal) of the domain Ω. The length of a vector $X\in\mathbb{C}^n$ ($\cong T_z\Omega$) with respect to this metric at $z\in\Omega$ is

$$\beta(z,X)=\beta_\Omega(z,X):=\sqrt{\sum_{i,j=1}^{n}T_{i\bar{j}}(z)X_i\bar{X}_j}. \tag{2}$$

The Bergman distance between two points $z,\zeta\in\Omega$ is

$$dist_\Omega(z,\zeta):=\inf_{\gamma\in S}\left\{\int_0^1\beta(\gamma(t),\gamma'(t))dt\right\}, \tag{3}$$

where S stands for the space of continuous piecewise $\mathcal{C}^1$ and parametrized by the interval [0, 1] curves with images in Ω, for which $\gamma(0)=z$, $\gamma(1)=\zeta$.

The completeness of the Bergman metric of Ω is the property that every Cauchy sequence with respect to $dist_\Omega$ has a limit point in Ω or equivalently, by the Hopf-Rinow theorem, that for any $z\in\Omega$, $z_0\in\partial\Omega$,

$$\lim_{\Omega\ni\zeta\to z_0}dist_\Omega(z,\zeta)=\infty,$$

where the limit is with respect to the Euclidean topology. The completeness of the Bergman metric of bounded domains in $\mathbb{C}^n$ has been studied extensively over the years (see [2,7,8,17–19,22,26,27] and [1,11] for qualitative results). In this paper we study the completeness of the following (closely related) Kähler metric.

$$\tilde{T}_{i\bar{j}}(z):=\left((n+1)T_{i\bar{j}}(z)+\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\log\det(T_{p\bar{q}}(z)_{p,q=1,\dots,n})\right). \tag{4}$$

By the well-known formula expressing the Ricci curvature of a Kähler metric this can be interpreted as

$$\tilde{T}_{i\bar{j}}(z)=(n+1)T_{i\bar{j}}(z)-Ric_{i\bar{j}}, \tag{5}$$

where $Ric_{i\bar{j}}$ is the Ricci tensor of the Bergman metric. It is well known that the Ricci curvature of the Bergman metric is bounded from above by $n+1$ (see [22]) and hence it follows that $\tilde{T}_{i\bar{j}}$ is positive definite and so it is indeed a metric. Such a metric was first considered in [15,21] (with a slightly different constant) and in [6]. It turned out that this metric is important in studying the so-called Bergman representative coordinates, see [13]. This metric also enjoys many of the properties of the Bergman metric. In particular it is invariant with respect to biholomorphic mappings, see [20].

As above, we define

$$\tilde{\beta}(z,X):=\sqrt{\sum_{i,j=1}^{n}\tilde{T}_{i\bar{j}}(z)X_i\bar{X}_j} \tag{6}$$

and

$$\tilde{dist}_{\Omega}(z,\zeta) := \inf_{\gamma \in S} \left\{ \int_0^1 \tilde{\beta}(\gamma(t), \gamma'(t))dt \right\}. \tag{7}$$

The completeness of $\tilde{T}_{i\bar{j}}$ is likewise defined as the property that every Cauchy sequence with respect to $\tilde{dist}_{\Omega}$ has a limit point in Ω or equivalently that for any $z \in \Omega$, $z_0 \in \partial\Omega$,

$$\lim_{\Omega \ni \zeta \to z_0} \tilde{dist}_{\Omega}(z,\zeta) = \infty. \tag{8}$$

Another important property that is shared with the Bergman metric is the fact that domains, which are complete with respect to $\tilde{T}_{i\bar{j}}$, are necessarily pseudoconvex (for the Bergman metric this follows by an old theorem by Bremermann [5], for $\tilde{T}_{i\bar{j}}$ the proof is virtually the same). For this reason we will restrict our attention to bounded pseudoconvex domains in $\mathbb{C}^n$ throughout the paper.

Clearly, if one of the metrics $T_{i\bar{j}}$, $\tilde{T}_{i\bar{j}}$ dominates some non-negative multiple of the other, then trivially its completeness follows from the completeness of the dominated metric. We have

Observation 1 *For $\Omega \subset\subset \mathbb{C}^n$*

a) if $T_{i\bar{j}}$ is complete and the Ricci curvature of the Bergman metric is bounded above by a constant C_1, with $C_1 < n+1$, then $\tilde{T}_{i\bar{j}}$ is complete;

b) if $\tilde{T}_{i\bar{j}}$ is complete and the Ricci curvature of the Bergman metric is bounded below by a constant C_2, then $T_{i\bar{j}}$ is complete.

In the special case of a strongly pseudoconvex domain Ω, Fefferman's asymptotic expansion of the Bergman kernel (see [14]) allows one to compute that the Ricci tensor of the Bergman metric tends to minus identity at the boundary of Ω (see also [24]). This, together with the fact that the Bergman metric is complete in strongly pseudoconvex domains, gives one immediately that $\tilde{T}_{i\bar{j}}$ is also complete.

Another instance, where the above observation can be used, is when Ω is a homogeneous bounded domain. Then the Bergman metric is Kähler-Einstein and hence its Ricci curvature is constant. The completeness of $\tilde{T}_{i\bar{j}}$ immediately follows.

In general, however, we cannot expect that the conditions on the Ricci curvature of the Bergman metric from the above observation will hold. In fact very few is known about the behavior of the Ricci curvature of the Bergman metric in general bounded domains. In [12] and [29] explicit examples of domains for which both conditions are violated were found. Moreover, such a domain can be hyperconvex (recall that hyperconvex domain is a domain for which there exists a bounded plurisubharmonic (in dimension 1 subharmonic) exhaustion function). This, together with the fact that bounded hyperconvex domains are complete with respect to the Bergman metric (see [2] and [17]), leads one to the following question: whether or not hyperconvex domains are complete with respect to $\tilde{T}_{i\bar{j}}$? A more general problem is to study in which classes of weakly pseudoconvex or even nonsmooth pseudoconvex domains is $\tilde{T}_{i\bar{j}}$ complete.

The examples from [12] and [29] enable one to look at the problems studied in this paper form yet another perspective. The completeness of $\tilde{T}_{i\bar{j}}$ is equivalent to the completeness of $T_{i\bar{j}} - \frac{1}{n+1}Ric_{i\bar{j}}$, which in certain cases may (presumably) be a gain in the study of the completeness of the Bergman metric.

2 Criterions for completeness and statement of the results

Denote by $L^2_h(\Omega) := L^2(\Omega) \cap \mathcal{O}(\Omega)$ the space of square-integrable holomorphic functions. We will benefit from the methods developed to study the completeness of the Bergman metric. The main tool for the study of completeness of the Bergman metric is the following criterion due to Kobayashi [22], see also [23].

Theorem 1 (Kobayashi) *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain. If for every function $f \in L^2_h(\Omega)$ and for every boundary point $z_0 \in \partial\Omega$ and for every sequence $\{z_s\}_{s=1}^{\infty} \subset \Omega$ of points in Ω with limit (in the Euclidean sense) z_0 there exists a subsequence $\{z_{s_k}\}_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} \frac{|f(z_{s_k})|^2}{K(z_{s_k}, z_{s_k})} = 0, \tag{9}$$

then the Bergman metric of Ω is complete.

This criterion has been modified by several authors (see e.g., [1]) and a version with weaker assumptions is

Theorem 2 (Błocki) *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain. If for every nonzero $f \in L^2_h(\Omega)$ and for every boundary point $z_0 \in \partial\Omega$ and for every sequence $\{z_s\}_{s=1}^{\infty} \subset \Omega$ of points in Ω with limit (in the Euclidean sense) z_0 there exists a subsequence $\{z_{s_k}\}_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} \frac{|f(z_{s_k})|^2}{K(z_{s_k}, z_{s_k})} < \|f\|^2_{L^2_h(\Omega)},$$

then the Bergman metric of Ω is complete.

We modify the methods of proof of Theorem 2 and obtain our

Theorem 3 *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain. If for every $n+1$- tuple of linearly independent $f_0, f_1, \ldots, f_n \in L^2_h(\Omega)$ and for every boundary point $z_0 \in \partial\Omega$ and for every sequence $\{z_s\}_{s=1}^{\infty} \subset \Omega$ of points in Ω with limit (in the Euclidean sense) z_0 there exists a subsequence $\{z_{s_k}\}_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} \left. \frac{\left| \det \begin{pmatrix} f_0(z) & \dots & f_n(z) \\ \frac{\partial f_0}{\partial z_1}(z) & \dots & \frac{\partial f_n}{\partial z_1}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial z_n}(z) & \dots & \frac{\partial f_n}{\partial z_n}(z) \end{pmatrix} \right|^2}{K^{n+1} \det \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K \right)} \right|_{z=z_{s_k}} \tag{10}$$

$$< \det \begin{pmatrix} \langle f_0, f_0 \rangle_{L^2_h(\Omega)} & \cdots & \langle f_n, f_0 \rangle_{L^2_h(\Omega)} \\ \vdots & \ddots & \vdots \\ \langle f_0, f_n \rangle_{L^2_h(\Omega)} & \cdots & \langle f_n, f_n \rangle_{L^2_h(\Omega)} \end{pmatrix},$$

then $\tilde{T}_{i\bar{j}}$ *is complete.*

Note that the right hand side of the above expression is the Gramian of the vectors $f_0, f_1, \dots, f_n$, which is positive, and hence a stronger assumption, which would also imply the completeness, is to require the limit in (10) to be 0.

To obtain this, we modify a construction of Lu Qi-Keng (see [25]), which goes as follows. If $\varphi_0, \varphi_1, \dots$ is a orthonormal basis of $L^2_h(\Omega)$, then one can embed holomorphically the domain Ω into the infinite dimensional Grassmannian of n- dimensional subspaces of ℓ^2, denoted by $\mathbb{F}(n, \infty)$, by means of

$$\Omega \ni z \to \left[\left. \begin{pmatrix} \varphi_0 \frac{\partial \varphi_1}{\partial z_1} - \varphi_1 \frac{\partial \varphi_0}{\partial z_1} & \varphi_0 \frac{\partial \varphi_2}{\partial z_1} - \varphi_2 \frac{\partial \varphi_0}{\partial z_1} & \varphi_1 \frac{\partial \varphi_2}{\partial z_1} - \varphi_2 \frac{\partial \varphi_1}{\partial z_1} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \varphi_0 \frac{\partial \varphi_1}{\partial z_n} - \varphi_1 \frac{\partial \varphi_0}{\partial z_n} & \varphi_0 \frac{\partial \varphi_2}{\partial z_n} - \varphi_2 \frac{\partial \varphi_0}{\partial z_n} & \varphi_1 \frac{\partial \varphi_2}{\partial z_n} - \varphi_2 \frac{\partial \varphi_1}{\partial z_n} & \cdots \end{pmatrix} \right|_z \right] \in \mathbb{F}(n, \infty), \tag{11}$$

where $[\cdot]$ is the equivalence relation between n- dimensional subspaces of ℓ^2 defining the points in the Grassmannian. This Grassmannian can further be embedded into some projective space by means of the Plücker embedding and eventually the pullback of the Fubini-Study metric by the composition of these two embeddings is exactly $\tilde{T}_{i\bar{j}}$ (see [13]). This approach has some significant disadvantages. The embedding is not independent of the basis, but the main problem is that, because partial derivatives of L^2 functions need not be L^2, the Grassmannian consists of subspaces of ℓ^2 and not $L^2_h(\Omega)$. Intuitively this is like a pointwise construction which due to the lack of uniformity is not enough to obtain our goals. Our new construction is also far simpler.

With the help of Theorem 3 we prove.

Theorem 4 *Bounded hyperconvex domains are complete with respect to* $\tilde{T}_{i\bar{j}}$.

In particular all pseudoconvex domains with Lipschitz boundaries, which are known to be hyperconvex (see [9]), are complete with respect to $\tilde{T}_{i\bar{j}}$.

3 Exterior products of Hilbert spaces

We begin with some basic facts about Hilbert spaces, which are not commonly seen in the theory of Bergman spaces. Let V be a complex vector space. We define the (algebraic) tensor product vector space $V \otimes V$ as the quotient vector space ${}^{U}/_{W}$ of some vector spaces U and W. Here U is the vector space generated by all pairs $(\alpha, \beta) \in V \times V$ as finite formal linear combinations with complex coefficients and W is the space generated in the same way by all elements of the following types

$$\begin{aligned}
&(\alpha+\beta, \gamma) - (\alpha, \gamma) - (\beta, \gamma);\\
&(\alpha, \beta+\gamma) - (\alpha, \beta) - (\alpha, \gamma);\\
&(a\alpha, \beta) - a(\alpha, \beta);\\
&(\alpha, a\beta) - a(\alpha, \beta),
\end{aligned}$$

where $\alpha, \beta, \gamma \in V$, $a \in \mathbb{C}$. Clearly W is a subspace of U. The tensor product $\alpha \otimes \beta$, which is a equivalence class, can be interpreted as the affine space $(\alpha, \beta) + W$. Now the wedge (or exterior) product $V \wedge V$ is defined as the quotient vector space ${}^{V\otimes V}/_{S}$, where $S \subset V \otimes V$ is the vector space generated by all elements of the type $\alpha \otimes \alpha$, where $\alpha \in V$. Again $\alpha \wedge \beta$ is a equivalence class, which can be interpreted as the affine space $\alpha \otimes \beta + S \subset V \otimes V$.

Let H be a separable Hilbert space, carrying the inner product $\langle \cdot, \cdot \rangle_H$. Now $H \wedge H$ makes sense at least as a vector space. This vector space $H \wedge H$ consists of all finite sums of the type $\sum_{i=1}^{m} a_i \alpha_i \wedge \beta_i$, where $a_i \in \mathbb{C}$, $\alpha_i, \beta_i \in H$, $m \in \mathbb{N}$. We endow this space with an inner product defined as follows. For elements of the type $\alpha \wedge \beta$ and $\gamma \wedge \delta$, where $\alpha, \beta, \gamma, \delta \in H$

$$\langle \alpha \wedge \beta, \gamma \wedge \delta \rangle_{H \wedge H} := \det \begin{pmatrix} \langle \alpha, \gamma \rangle_H & \langle \alpha, \delta \rangle_H \\ \langle \beta, \gamma \rangle_H & \langle \beta, \delta \rangle_H \end{pmatrix}. \tag{12}$$

After defining the inner product on such vectors we extend it on the whole vector space $H \wedge H$ by linearity. Now we perform the completion of $H \wedge H$ with respect to $\langle \cdot, \cdot \rangle_{H \wedge H}$, that is we allow not only finite but also countable combinations $\sum_{i=1}^{\infty} a_i \alpha_i \wedge \beta_i$, obeying the natural restriction that $\sum_{i=1}^{\infty} |a_i|^2 < \infty$. By abusing notation, we agree to call this completion also $H \wedge H$. The inner product also extends to the completed vector space and again by abusing notation we call the extension $\langle \cdot, \cdot \rangle_{H \wedge H}$. Now it is easy to see that $(H \wedge H, \langle \cdot, \cdot \rangle_{H \wedge H})$ is a Hilbert space. It is also easy to see that this Hilbert space is separable.

Likewise if we take $n+1$ copies of a Hilbert space F, we can define the Hilbert space $(F \wedge \cdots \wedge F, \langle \cdot, \cdot \rangle_{F \wedge \cdots \wedge F})$, which is the completion of the vector space $F \wedge \cdots \wedge F$ with respect to the inner product, which is the linear extension of

$$\langle \alpha_0 \wedge \cdots \wedge \alpha_n, \beta_0 \wedge \cdots \wedge \beta_n \rangle_{F \wedge \cdots \wedge F} := \det \begin{pmatrix} \langle \alpha_0, \beta_0 \rangle_F & \cdots & \langle \alpha_0, \beta_n \rangle_F \\ \vdots & \ddots & \vdots \\ \langle \alpha_n, \beta_0 \rangle_F & \cdots & \langle \alpha_n, \beta_n \rangle_F \end{pmatrix}. \tag{13}$$

It is a matter of algebraic manipulations to see that the continuous dual space of $F \wedge \cdots \wedge F$ satisfies

$$(F \wedge \cdots \wedge F)' \cong F' \wedge \cdots \wedge F'. \tag{14}$$

A proof of this fact can be found in [4].

A element $\alpha \in F \wedge \cdots \wedge F$ which can be represented as $\alpha = \alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \alpha_n$, for some $\alpha_i \in F, i = 0, \ldots, n$ will be called decomposable (the terms pure, monomial, simple and completely reducible are also frequent in the literature). Clearly not all elements of $F \wedge \cdots \wedge F$ are decomposable. There is a criterion for determining whether a nonzero vector is decomposable or not, known as Plücker (or Plücker-Grassmann) conditions. To introduce it we need more notation. Let J be a s- tuple of natural numbers $j_1 < \cdots < j_s$. We denote by e_J the vector $e_{j_1} \wedge \cdots \wedge e_{j_s}$, where e_j is a fixed orthonormal basis of a separable Hilbert space E. Clearly $e_J \in E \wedge \cdots \wedge E$, where the exterior product is taken s times, and moreover the vectors e_J, for all possible s-tuples J of pairwise distinct natural numbers, form a orthonormal basis of $E \wedge \cdots \wedge E$. We can therefore expand a vector $\alpha \in E \wedge \cdots \wedge E$ as $\alpha = \sum_J a_J e_J$, where $a_J = \langle \alpha, e_J \rangle_{E \wedge \cdots \wedge E} \in \mathbb{C}$. Now a nonzero vector α is decomposable if and only if for all $I \subset \mathbb{N}^{s-1}$ and for all $L \subset \mathbb{N}^{s+1}$, both I and L without recurring elements, such that $I \cap L = \emptyset$, the following equality holds

$$\sum_{i \in L} \rho_{I,L,i} a_{I \cup \{i\}} a_{L \setminus \{i\}} = 0, \tag{15}$$

where $\rho_{I,L,i} = 1$ if $\sharp\{j \in L : j < i\} \equiv \sharp\{j \in I : j < i\} \pmod 2$ and $\rho_{I,L,i} = -1$ otherwise. Also in the index notation $a_{I \cup \{i\}}$ (respectively $a_{L \setminus \{i\}}$) it should be clarified that the elements of the sets $I \cup \{i\}$ (respectively $L \setminus \{i\}$) are ordered in a increasing fashion. For a proof see [16], Chapter 22. Actually in [16] only the finite-dimensional case is considered, however, one should take the continuous dual space instead of the algebraic dual space and the argument goes mutatis-mutandis.

Lemma 1 *If a sequence $\{\alpha_i\}_{i=1}^{\infty}$ of unit vectors in $F \wedge \cdots \wedge F$ has a limit $\alpha \in F \wedge \cdots \wedge F$ in the norm topology and moreover each α_i is of the form $b_i \alpha_{i0} \wedge \alpha_{i1} \wedge \cdots \wedge \alpha_{in}$, where $b_i \in \mathbb{C}, \alpha_{ij} \in F, j = 0, \ldots, n, i = 1, \ldots,$ then also α is a unit vector of the form $b\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \alpha_n$, for some $b \in \mathbb{C}, \alpha_j \in F, j = 0, \ldots, n$ (that is the limit is a decomposable vector).*

First observe that it is not true in general that if a sequence $f_s \wedge g_s$, $f_s, g_s \in E$, for some Hilbert space E, has a limit in $E \wedge E$ then necessarily f_s and g_s both have limits in E and the simplest counterexample is just $f_s = sf$, $g_s = \frac{1}{s} g$, for some fixed $f, g \in E$.

Proof We expand the elements of the sequence, as well as the limit, into

$$\alpha_i = \sum_J a_J^i e_J, \alpha = \sum_J a_J e_J.$$

Since $\|\alpha_i - \alpha\|_{F\wedge\cdots\wedge F} \to 0$, it follows that $|a^i_J - a_J| \to 0$. By the assumption and the Plücker relations (15) we have

$$\sum_{s\in L} \rho_{I,L,s} a^i_{I\cup\{s\}} a^i_{L\setminus\{s\}} = 0,$$

for all subsets $I \subset \mathbb{N}^n$, $L \subset \mathbb{N}^{n+2}$, without repetitions, such that $I \cap L = \emptyset$. Now it is obvious that also

$$\sum_{s\in L} \rho_{I,L,s} a_{I\cup\{s\}} a_{L\setminus\{s\}} = 0.$$

For more on these items one should consult [4], Chapter 5, §3,4, where tensor and exterior products of Hilbert spaces are explicitly considered, [3], Chapter 3, for more results but in a more abstract algebraic setting and also [16], Chapter 22, where the concepts of decomposable vectors and tests for decomposability are very clearly presented, however, only in finite dimensions.

4 The construction

In our case F will be $L^2_h(\Omega)'$ - the Hilbert space which is the continuous dual space of $L^2_h(\Omega)$ and so $F \wedge \cdots \wedge F = L^2_h(\Omega)' \wedge \cdots \wedge L^2_h(\Omega)'$ can be identified with the Hilbert space of multilinear antisymmetric continuous mappings (forms) from $L^2_h(\Omega) \times \cdots \times L^2_h(\Omega)$ to $\mathbb{C}$. Actually the forms are defined on $L^2_h(\Omega) \wedge \cdots \wedge L^2_h(\Omega)$ rather than on $L^2_h(\Omega) \times \cdots \times L^2_h(\Omega)$ but the definition can be extended in a obvious and canonical way. A element α of the Hilbert space $F \wedge \cdots \wedge F$ can be written down as a linear combination of the form

$$\alpha = \sum_{i=1}^{\infty} a_i \alpha_{i0} \wedge \alpha_{i1} \wedge \cdots \wedge \alpha_{in},$$

where $\{a_i\}_{i=1}^{\infty} \in \ell^2, \alpha_{ij} \in L^2_h(\Omega)', j = 0, \ldots, n, i = 1, \ldots$. The aforementioned identification with a multilinear antisymmetric form is realized by first identifying elements of the type $\alpha_{i0} \wedge \alpha_{i1} \wedge \cdots \wedge \alpha_{in}$ by

$$\begin{aligned} \alpha_{i0} \wedge \alpha_{i1} \wedge \cdots \wedge \alpha_{in} &\cong L^2_h(\Omega) \times \cdots \times L^2_h(\Omega) \ni (f_0, \ldots, f_n) \\ &\to \det \begin{pmatrix} \alpha_{i0}(f_0) & \cdots & \alpha_{i0}(f_n) \\ \vdots & \ddots & \vdots \\ \alpha_{in}(f_0) & \cdots & \alpha_{in}(f_n) \end{pmatrix} \\ &= \alpha_{i0} \wedge \alpha_{i1} \wedge \cdots \wedge \alpha_{in}(f_0, \ldots, f_n) \in \mathbb{C} \end{aligned} \tag{16}$$

and extending it linearly on the whole $L^2_h(\Omega)' \wedge \cdots \wedge L^2_h(\Omega)'$ afterwards. This is consistent with the introduced inner product and hence the correspondence is clearly a isomorphism of Hilbert spaces.

By the Cauchy estimates the following linear mappings are continuous

$$i(z) : L^2_h(\Omega) \ni f \to f(z) \in \mathbb{C}, \tag{17}$$

$$j_1(z) : L^2_h(\Omega) \ni f \to \frac{\partial f}{\partial z_1}(z) \in \mathbb{C}, \tag{18}$$

$$\dots$$

$$j_n(z) : L^2_h(\Omega) \ni f \to \frac{\partial f}{\partial z_n}(z) \in \mathbb{C}. \tag{19}$$

By the Riesz theorem for every $l \in L^2_h(\Omega)'$ there is a unique $l' \in L^2_h(\Omega)$ such that $l(\cdot) = \langle \cdot, l' \rangle_{L^2_h(\Omega)}$. Moreover, $\langle k, l \rangle_{L^2_h(\Omega)'} = \overline{\langle k', l' \rangle}_{L^2_h(\Omega)} = \langle l', k' \rangle_{L^2_h(\Omega)}$. In our case one can easily check by using the reproducing property of the Bergman kernel that

$$i(z)' = K(\cdot, z) \in L^2_h(\Omega), \tag{20}$$

$$j_s(z)' = \left. \frac{\partial K(\cdot, \zeta)}{\partial \bar{\zeta}_s} \right|_{\zeta = z} \in L^2_h(\Omega), s = 1, \dots, n. \tag{21}$$

Let $\mathbb{P}(F \wedge \cdots \wedge F)$ be the projectivization of the Hilbert space $F \wedge \cdots \wedge F$, that is the quotient space ${}^{F \wedge \cdots \wedge F}/_{\sim}$ with respect to the following (projective) equivalence relation. For $u, v \in F \wedge \cdots \wedge F \setminus \{0\}$ we have $u \sim v$ if and only if $u = cv$, for some $c \in \mathbb{C} \setminus \{0\}$. For more on projectivizations of infinite dimensional Hilbert spaces see [22]. We embed Ω into the projective space $\mathbb{P}(F \wedge \cdots \wedge F)$ by the mapping

$$\Omega \ni z \to [i(z) \wedge j_1(z) \wedge j_2(z) \wedge \cdots \wedge j_n(z)] \in \mathbb{P}(F \wedge \cdots \wedge F), \tag{22}$$

where $[\cdot]$ is the equivalence class with respect to $\sim$. This is a holomorphic embedding.

Lemma 2 *The value of $i(z) \wedge j_1(z) \wedge \cdots \wedge j_n(z)$, interpreted as an antisymmetric multilinear form on $L^2_h(\Omega) \times \cdots \times L^2_h(\Omega)$, at the point $(f_0, f_1, \dots, f_n) \in L^2_h(\Omega) \times \cdots \times L^2_h(\Omega)$ is*

$$\det \begin{pmatrix} f_0(z) & \dots & f_n(z) \\ \frac{\partial f_0}{\partial z_1}(z) & \dots & \frac{\partial f_n}{\partial z_1}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial z_n}(z) & \dots & \frac{\partial f_n}{\partial z_n}(z) \end{pmatrix}.$$

The proof is a immediate consequence of (16), (17), (18) and (19).

Lemma 3 *The square of the norm of $i(z) \wedge j_1(z) \wedge \cdots \wedge j_n(z)$ in $F \wedge \cdots \wedge F$ equals*

$$\|i(z) \wedge j_1(z) \wedge \cdots \wedge j_n(z)\|^2_{F \wedge \cdots \wedge F} = K^{n+1} \det \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K \right) \Bigg|_z .$$

Proof By (13), the Riesz theorem, (20), (21) and the reproducing property of the Bergman kernel we have

$$
\begin{aligned}
&\|i(z) \wedge j_1(z) \wedge \cdots \wedge j_n(z)\|^2_{F\wedge\cdots\wedge F}\\
&= \langle i(z) \wedge j_1(z) \wedge \cdots \wedge j_n(z), i(z) \wedge j_1(z) \wedge \cdots \wedge j_n(z)\rangle_{F\wedge\cdots\wedge F}\\
&= \det \begin{pmatrix} \langle i(z), i(z)\rangle_{L^2_h(\Omega)'} & \langle i(z), j_1(z)\rangle_{L^2_h(\Omega)'} & \ldots & \langle i(z), j_n(z)\rangle_{L^2_h(\Omega)'} \\ \langle j_1(z), i(z)\rangle_{L^2_h(\Omega)'} & \langle j_1(z), j_1(z)\rangle_{L^2_h(\Omega)'} & \ldots & \langle j_1(z), j_n(z)\rangle_{L^2_h(\Omega)'} \\ \vdots & \vdots & \ddots & \vdots \\ \langle j_n(z), i(z)\rangle_{L^2_h(\Omega)'} & \langle j_n(z), j_1(z)\rangle_{L^2_h(\Omega)'} & \ldots & \langle j_n(z), j_n(z)\rangle_{L^2_h(\Omega)'} \end{pmatrix}\\
&= \det \begin{pmatrix} \langle K, K\rangle_{L^2_h(\Omega)} & \left\langle \frac{\partial K}{\partial \bar\zeta_1}, K\right\rangle_{L^2_h(\Omega)} & \ldots & \left\langle \frac{\partial K}{\partial \bar\zeta_n}, K\right\rangle_{L^2_h(\Omega)} \\ \left\langle K, \frac{\partial K}{\partial \bar\zeta_1}\right\rangle_{L^2_h(\Omega)} & \left\langle \frac{\partial K}{\partial \bar\zeta_1}, \frac{\partial K}{\partial \bar\zeta_1}\right\rangle_{L^2_h(\Omega)} & \ldots & \left\langle \frac{\partial K}{\partial \bar\zeta_n}, \frac{\partial K}{\partial \bar\zeta_1}\right\rangle_{L^2_h(\Omega)} \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle K, \frac{\partial K}{\partial \bar\zeta_n}\right\rangle_{L^2_h(\Omega)} & \left\langle \frac{\partial K}{\partial \bar\zeta_1}, \frac{\partial K}{\partial \bar\zeta_n}\right\rangle_{L^2_h(\Omega)} & \ldots & \left\langle \frac{\partial K}{\partial \bar\zeta_n}, \frac{\partial K}{\partial \bar\zeta_n}\right\rangle_{L^2_h(\Omega)} \end{pmatrix}\Bigg|_z\\
&= \det \begin{pmatrix} K & \frac{\partial K}{\partial \bar\zeta_1} & \ldots & \frac{\partial K}{\partial \bar\zeta_n} \\ \frac{\partial K}{\partial \zeta_1} & \frac{\partial^2 K}{\partial \zeta_1 \partial \bar\zeta_1}, & \ldots & \frac{\partial^2 K}{\partial \zeta_1 \partial \bar\zeta_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial K}{\partial \zeta_n} & \frac{\partial^2 K}{\partial \zeta_n \partial \bar\zeta_1}, & \ldots & \frac{\partial^2 K}{\partial \zeta_n \partial \bar\zeta_n} \end{pmatrix}\Bigg|_z .
\end{aligned}
$$

By a well known formula (see e.g., [24]) the last expression equals

$$K^{n+1} \det \left(\frac{\partial^2}{\partial z_i \partial \bar z_j} \log K \right)\Bigg|_z .$$

Theorem 5 *The embedding (22) is isometric, that is, the pullback of the Fubini-Study metric on $\mathbb{P}(F \wedge \cdots \wedge F)$ is exactly the metric $\tilde T_{i\bar j}$.*

Proof First recall that the Fubini-Study metric on a projectivization $\mathbb{P}(E)$ of a Hilbert space E at the point $[\zeta] \in \mathbb{P}(E)$ has the following metric tensor

$$\mathcal{FS}_{p\bar q} := \frac{\partial^2}{\partial \zeta_p \partial \bar\zeta_q} \log \|\zeta\|^2_E.$$

Note that the definition does not depend ot the choice of the (nonzero) representative $\zeta \in [\zeta]$. Let the image of the point $z \in \Omega$ be $[i(z) \wedge j_1(z) \wedge j_2(z) \wedge \cdots \wedge j_n(z)] = [\zeta]$. By Lemma (3) the pullback of the Fubini-Study metric is the metric with metric tensor

$$
\begin{aligned}
&[i(\cdot) \wedge j_1(\cdot) \wedge j_2(\cdot) \wedge \cdots \wedge j_n(\cdot)]^*(\mathcal{FS}_{p\bar{q}}) \\
&= \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \left(K^{n+1} \det \left(\frac{\partial^2}{\partial z_r \partial \bar{z}_s} \log K \right) \Big|_z \right) \\
&= (n+1) T_{i\bar{j}}(z) + \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \left(\det T_{r\bar{s}}(z)_{r,s=1,\ldots,n} \right) = \tilde{T}_{i\bar{j}}
\end{aligned}
$$

For more on the Fubini-Study metric on projectivizations of Hilbert spaces and related items see [22].

Theorem 6 *The following equality holds*

$$
\begin{aligned}
&K^{n+1} \det \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K \right) \Big|_z \\
&= \sup_{\substack{(f_0,\ldots,f_n) \in L^2_h(\Omega) \times \cdots \times L^2_h(\Omega): \\ f_0 \wedge \cdots \wedge f_n \neq 0}} \frac{\left| \det \begin{pmatrix} f_0(z) & \ldots & f_n(z) \\ \frac{\partial f_0}{\partial z_1}(z) & \ldots & \frac{\partial f_n}{\partial z_1}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial z_n}(z) & \ldots & \frac{\partial f_n}{\partial z_n}(z) \end{pmatrix} \right|^2}{\det \begin{pmatrix} \langle f_0, f_0 \rangle_{L^2_h(\Omega)} & \cdots & \langle f_0, f_n \rangle_{L^2_h(\Omega)} \\ \vdots & \ddots & \vdots \\ \langle f_n, f_0 \rangle_{L^2_h(\Omega)} & \cdots & \langle f_n, f_n \rangle_{L^2_h(\Omega)} \end{pmatrix}}.
\end{aligned}
$$

Proof We have the isometry

$$
\left(L^2_h(\Omega) \wedge \cdots \wedge L^2_h(\Omega) \right)' \cong L^2_h(\Omega)' \wedge \cdots \wedge L^2_h(\Omega)'.
$$

As usual the norm of a linear functional is

$$
\|\alpha\|_{\left(L^2_h(\Omega) \wedge \cdots \wedge L^2_h(\Omega)\right)'} = \sup \frac{|\alpha(f)|}{\|f\|_{L^2_h(\Omega) \wedge \cdots \wedge L^2_h(\Omega)}},
$$

where the supremum is taken over all nonzero $f \in L^2_h(\Omega) \wedge \cdots \wedge L^2_h(\Omega)$. By the Riesz theorem the supremum is attained at the vector $f = \alpha'$. When α is decomposable, we use the fact that α' is also decomposable. In fact if $\alpha = \alpha_0 \wedge \cdots \wedge \alpha_n$ then $\alpha' = \alpha_0' \wedge \cdots \wedge \alpha_n'$. By Lemma 3, the decomposability of $i(z) \wedge j_1(z) \wedge \cdots \wedge j_n(z)$ and Lemma 2 we have

$$
\begin{aligned}
&K^{n+1} \det \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K \right) \Big|_z = \|i(z) \wedge j_1(z) \wedge \cdots \wedge j_n(z)\|^2_{L^2_h(\Omega)' \wedge \cdots \wedge L^2_h(\Omega)'} \\
&= \|i(z) \wedge j_1(z) \wedge \cdots \wedge j_n(z)\|^2_{\left(L^2_h(\Omega) \wedge \cdots \wedge L^2_h(\Omega)\right)'}
\end{aligned}
$$

$$= \sup_{0\neq f_0\wedge\cdots\wedge f_n\in L_h^2(\Omega)\wedge\cdots\wedge L_h^2(\Omega)} \frac{|i(z)\wedge j_1(z)\wedge\cdots\wedge j_n(z)(f_0,\ldots,f_n)|^2}{\|f_0\wedge\cdots\wedge f_n\|^2_{L_h^2(\Omega)\wedge\cdots\wedge L_h^2(\Omega)}}$$

$$= \sup_{\substack{(f_0,\ldots,f_n)\in L_h^2(\Omega)\times\cdots\times L_h^2(\Omega):\\ f_0\wedge\cdots\wedge f_n\neq 0}} \frac{\left|\det\begin{pmatrix} f_0(z) & \ldots & f_n(z)\\ \frac{\partial f_0}{\partial z_1}(z) & \ldots & \frac{\partial f_n}{\partial z_1}(z)\\ \vdots & \ddots & \vdots\\ \frac{\partial f_0}{\partial z_n}(z) & \ldots & \frac{\partial f_n}{\partial z_n}(z)\end{pmatrix}\right|^2}{\det\begin{pmatrix}\langle f_0,f_0\rangle_{L_h^2(\Omega)} & \cdots & \langle f_0,f_n\rangle_{L_h^2(\Omega)}\\ \vdots & \ddots & \vdots\\ \langle f_n,f_0\rangle_{L_h^2(\Omega)} & \cdots & \langle f_n,f_n\rangle_{L_h^2(\Omega)}\end{pmatrix}}.$$

5 Proofs of the Theorems and open problems

Proof of Theorem 3 We proceed as in [22]. Suppose that the metric $\tilde{T}_{i\bar{j}}$ is not complete in Ω. We choose a Cauchy (with respect to $\tilde{T}_{i\bar{j}}$) sequence $\{z_s\}_{s=1}^{\infty}\subset\Omega$ which has no convergent (again with respect to $\tilde{T}_{i\bar{j}}$) subsequence. Now we use Theorem 5 and embed holomorphically and isometrically Ω with the metric $\tilde{T}_{i\bar{j}}$ into $\mathbb{P}(F\wedge\cdots\wedge F)$ with the Fubini-Study metric by the mapping (22). The image sequence $[i(z_s)\wedge j_1(z_s)\wedge\cdots\wedge j_n(z_s)]$ is also a Cauchy sequence with respect to the Fubini-Study metric, because isometries do not increase distance. The space $\mathbb{P}(F\wedge\cdots\wedge F)$ is, however, complete and hence the image sequence has a convergent subsequence (with respect to the Fubini-Study metric) $[i(z_{s_k})\wedge j_1(z_{s_k})\wedge\cdots\wedge j_n(z_{s_k})]$ with limit $f\in\mathbb{P}(F\wedge\cdots\wedge F)$. This means that also the unit vectors

$$e^{i\theta_k}\frac{i(z_{s_k})\wedge j_1(z_{s_k})\wedge\cdots\wedge j_n(z_{s_k})}{\|i(z_{s_k})\wedge j_1(z_{s_k})\wedge\cdots\wedge j_n(z_{s_k})\|}\in F\wedge\cdots\wedge F,$$

which represent the above classes, converge for a proper choice of $\theta_k\in[0,2\pi)$ in $F\wedge\cdots\wedge F$ to some α, which represents the class f. Now α is a unit vector and moreover, by Lemma 1, $\alpha=\alpha_0\wedge\cdots\wedge\alpha_n$, for some $\alpha_0,\ldots,\alpha_n\in L_h^2(\Omega)'$. The vector α is nonzero and hence the vectors α_j, $j=0,\ldots,n$, are linearly independent in $L_h^2(\Omega)'$. For each α_j take its Hilbert dual $f_j\in L_h^2(\Omega)$. Clearly also the vectors f_j, $j=0,\ldots,n$, are linearly independent. Now by Lemmas 2, 3 and the formula (16)

$$\left.\frac{\left|\det\begin{pmatrix} f_0(z) & \ldots & f_n(z)\\ \frac{\partial f_0}{\partial z_1}(z) & \ldots & \frac{\partial f_n}{\partial z_1}(z)\\ \vdots & \ddots & \vdots\\ \frac{\partial f_0}{\partial z_n}(z) & \ldots & \frac{\partial f_n}{\partial z_n}(z)\end{pmatrix}\right|^2}{K^{n+1}\det\left(\frac{\partial^2}{\partial z_i\partial\bar{z}_j}\log K\right)}\right|_{z=z_{s_k}} = \frac{\left|i(z_{s_k})\wedge j_1(z_{s_k})\wedge\cdots\wedge j_n(z_{s_k})(f_0,\ldots,f_n)\right|^2}{\|i(z_{s_k})\wedge j_1(z_{s_k})\wedge\cdots\wedge j_n(z_{s_k})\|^2}$$

$$\to |\alpha_0 \wedge \cdots \wedge \alpha_n(f_0, \ldots, f_n)|^2 = \det \begin{pmatrix} \langle f_0, f_0 \rangle_{L_h^2(\Omega)} & \cdots & \langle f_n, f_0 \rangle_{L_h^2(\Omega)} \\ \vdots & \ddots & \vdots \\ \langle f_0, f_n \rangle_{L_h^2(\Omega)} & \cdots & \langle f_n, f_n \rangle_{L_h^2(\Omega)} \end{pmatrix}.$$

This contradicts the assumptions of Theorem 3.

Recall that the pluricomplex Green function of the bounded domain $\Omega \subset \mathbb{C}^n$ with logarithmic singularity at $z \in \Omega$ is the function

$$G_\Omega(\cdot, z) := \sup_{\varphi \in PSH(\Omega)} \left\{ \varphi(\cdot) : \varphi < 0, \limsup_{\zeta \to z} (\varphi(\zeta) - \log|\zeta - z|) < \infty \right\},$$

where $PSH(\Omega)$ is the space of plurisubharmonic functions on Ω. The function $G_\Omega(\cdot, z)$ is plurisubharmonic and negative in Ω. We will need a lemma.

Lemma 4 *For every bounded pseudoconvex domain Ω there exists a constant $C > 0$, such that for every $f \in L_h^2(\Omega)$ and a given $z \in \Omega$, one can find $\tilde{f} \in L_h^2(\Omega)$, such that $f(z) = \tilde{f}(z)$ and $\frac{\partial f}{\partial z_j}(z) = \frac{\partial \tilde{f}}{\partial z_j}(z)$ and moreover*

$$\int_\Omega |\tilde{f}|^2 d\lambda \leq C \int_{\{G_\Omega(\cdot, z) < -1\}} |f|^2 d\lambda,$$

where $d\lambda$ is the Lebesgue measure.

This is a simpler version of Lemma 4.2 in [10]. The proof uses Hörmander's estimates for the $\bar{\partial}$-equation and can be found in [10]. The constant C can be chosen to be $1 + e^{4n+7+(max_\Omega |z|)^2}$.

Proof of Theorem 4 At the point z_{s_k} we construct the corresponding functions $\tilde{f}_j$ for each of the functions f_j, $j = 0, \ldots, n$ from Lemma 4. Then by Theorem 6, Hadamard's inequality and Lemma 4 we have

$$\left. \frac{\left| \det \begin{pmatrix} f_0(z) & \ldots & f_n(z) \\ \frac{\partial f_0}{\partial z_1}(z) & \ldots & \frac{\partial f_n}{\partial z_1}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial z_n}(z) & \ldots & \frac{\partial f_n}{\partial z_n}(z) \end{pmatrix} \right|^2}{K^{n+1} \det \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K \right)} \right|_{z=z_{s_k}} = \left. \frac{\left| \det \begin{pmatrix} \tilde{f}_0(z) & \ldots & \tilde{f}_n(z) \\ \frac{\partial \tilde{f}_0}{\partial z_1}(z) & \ldots & \frac{\partial \tilde{f}_n}{\partial z_1}(z) \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{f}_0}{\partial z_n}(z) & \ldots & \frac{\partial \tilde{f}_n}{\partial z_n}(z) \end{pmatrix} \right|^2}{K^{n+1} \det \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K \right)} \right|_{z=z_{s_k}}$$

$$\leq \det \begin{pmatrix} \langle \tilde{f}_0, \tilde{f}_0 \rangle_{L_h^2(\Omega)} & \cdots & \langle \tilde{f}_0, \tilde{f}_n \rangle_{L_h^2(\Omega)} \\ \vdots & \ddots & \vdots \\ \langle \tilde{f}_n, \tilde{f}_0 \rangle_{L_h^2(\Omega)} & \cdots & \langle \tilde{f}_n, \tilde{f}_n \rangle_{L_h^2(\Omega)} \end{pmatrix} \leq \|\tilde{f}_0\|^2_{L_h^2(\Omega)} \cdots \|\tilde{f}_0\|^2_{L_h^2(\Omega)}$$

$$\leq C^{n+1} \int_{\{G_\Omega(\cdot, z_{s_k}) < -1\}} |f_0|^2 d\lambda \ldots \int_{\{G_\Omega(\cdot, z_{s_k}) < -1\}} |f_n|^2 d\lambda \to 0,$$

because each $f_j \in L^2_h(\Omega)$ and the volume of $\{G_\Omega(\cdot, z) < -1\}$ goes to 0 as $k \to \infty$ in bounded hyperconvex domains, see [2] or [17].

We do not know whether or not there exist domains for which one of the metrics $T_{i\bar{j}}$, $\tilde{T}_{i\bar{j}}$ is complete and the other is not.

Despite the suggestion in [22] that domains in which the Bergman metric is complete should satisfy the conditions in Theorem 1, Zwonek in [28] constructed a domain for which the Bergman metric is complete and the limit in (9) is not zero. This means that the Kobayashi criterion (Theorem 1) is not a if and only if statement. As noted in [1] it is not known whether or not the modified version Theorem 2 is a if and only if statement. Likewise we do not know whether or not the criterion in Theorem 3 is a if and only if statement. We do not know this even if the limit in (10) is assumed to be 0.

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